NASA TT F-10,141

NASA TT F-10,141

PROBLEMS OF OSCILLATIONS OF A FLUID SUBJECTED TO SURFACE TENSION FORCES

N.N.Moiseyev and F.L.Chernous'ko

Translation of "Zadachi kolebaniy zhidkosti, podverzhennoy silam poverkhnostnogo natyazheniya".
Zhurnal Vychislitel'noy Matematiki i Matematicheskoy Fiziki, Vol.5, No.6, pp.1071-1095, 1965.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON MAY 1966

NASA TT F-10,141

PROBLEMS OF OSCILLATIONS OF A FLUID SUBJECTED TO SURFACE TENSION FORCES */1071

N.N.Moiseyev and F.L.Chernous'ko

Discussion of certain problems arising in the theory of the behavior of fluids under conditions of weightlessness and in weak gravitational fields. The problem of small linear oscillations of an ideal fluid is formulated, and the conditions of solvability of this problem and the properties of its spectral structure are discussed. It is observed that, although the problem of small oscillations theoretically involves no special difficulties, practically no effective solutions to it are known. It is shown that the problem of small oscillations of an ideal fluid in the presence of surface tension can be effectively solved when the surface tension is small in comparison with mass forces. The equilibrium and natural oscillations of a heavy ideal fluid in a vessel are studied under this assumption.

Recently, various problems of the behavior of fluids under conditions of weightlessness and in weak gravitational fields have become topical. The present article is devoted to two aspects of the oscillation theory. The first two Sections deal with the formulation of the problem of small oscillations of fluids, discuss the questions of the solvability of problems of this theory, and establish the properties of the spectra. The other Sections deal with the

* Numbers in the margin indicate pagination in the original foreign text.

oscillations of fluids under various simplifying assumptions.

Section 1. Formulation of the Problem of the Theory of Oscillations of an Ideal Fluid

1. If the intensity of volume forces $F = \nabla U$ is low, the forces of surface tension become of decisive significance in problems of the dynamics of an ideal fluid. The motion of an ideal incompressible fluid is described by Euler's and continuity equations

$$\frac{dv}{dt} = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}F, \tag{1.1}$$

$$\nabla v = 0. \tag{1.2}$$

where v is the velocity, ρ is the fluid density, p is the pressure, and t is the time.

At an immobile wall (surface Σ) the condition of non-leakage

$$v n_\Sigma = 0, \tag{1.3}$$

must be satisfied, where n_Σ is the unit vector of the normal to Σ. At the free surface (surface S), the condition of thermodynamic equilibrium

$$p = \sigma K + \text{const}, \qquad K = 1/R_1 + 1/R_2. \tag{1.4}$$

must be satisfied, where K is the double mean curvature of the surface S, while R_1 and R_2 are its principal radii of curvature, and σ is the coefficient of surface tension. At points along the line of intersection between the surfaces S and Σ (contour Γ) the normals to these surfaces form a constant angle γ of contact, which depends solely on the material of shell Σ and on the properties of the fluid. The equation of the free surface and the shell will be speci- /1072
fied as $\Phi_s(x, y, z, t) = 0$ and $\Phi_\Sigma(x, y, z) = 0$. Then, this condition can be written as

$$\frac{\partial \Phi_\Sigma}{\partial x}\frac{\partial \Phi_S}{\partial x} + \frac{\partial \Phi_\Sigma}{\partial y}\frac{\partial \Phi_S}{\partial y} + \frac{\partial \Phi_\Sigma}{\partial z}\frac{\partial \Phi_S}{\partial z} = N\cos\gamma, \tag{1.5}$$

where $N = |\nabla\Phi_s| \; |\nabla\Phi_\Sigma|$ is the normalizing factor. Hereafter, the equation of the free boundary will often be written in the form of $z = Z(x, y, t)$.

In this case, the condition (1.5) will become

$$\frac{\partial \Phi_z}{\partial z} - \frac{\partial \Phi_z}{\partial x}\frac{\partial Z}{\partial x} - \frac{\partial \Phi_z}{\partial y}\frac{\partial Z}{\partial y} = N \cos\gamma. \tag{1.6}$$

In order to make the problem definite, the kinematic relation

$$\frac{\partial \Phi_s}{\partial t} + N_1 v_n = 0, \tag{1.7}$$

must be added to the above conditions. Here, $v_n = (\mathbf{v}, \mathbf{n}_s)$ is the projection of the velocity vector onto the normal to the surface S; N_1 is the normalizing factor:

$$N_1 = \left[\left(\frac{\partial \Phi_s}{\partial x}\right)^2 + \left(\frac{\partial \Phi_s}{\partial y}\right)^2 + \left(\frac{\partial \Phi_s}{\partial z}\right)^2\right]^{1/2}.$$

2. First, let us consider a static problem. If $v \equiv 0$, then $p = U + \text{const}$ and the condition (1.4) will assume the form

$$U - \sigma K = \text{const}. \tag{1.8}$$

The constant in eq.(1.8) can be taken as zero without restriction as to generality. The expression $U - \sigma K$ results from the action of some nonlinear differential operator on the function $Z(x, y)$. Hence, the form of the free surface in equilibrium position satisfies the nonlinear partial differential equation (1.8) and the boundary condition (1.6). In addition, the free surface [function $Z(x, y)$] must satisfy the condition of isoperimetricity: volume of the fluid given.

Generally, this is a difficult problem and forms a separate domain of investigation. Here we will dwell only on one elementary case (reducing to the calculation of the roots of a transcendental equation).

Let us assume that mass forces are absent; then the free surface is a surface with constant curvature. Let us further assume that the region occupied by the fluid has an axis of symmetry and that its volume is V. Thus, one of the possible solutions of the boundary-value problem [eqs.(1.8), (1.6)] will be a sphere whose line of intersection with the surface Σ is a two-dimensional curve (cf. Fig.1a, giving the notation) and whose center is located at the axis of symmetry.

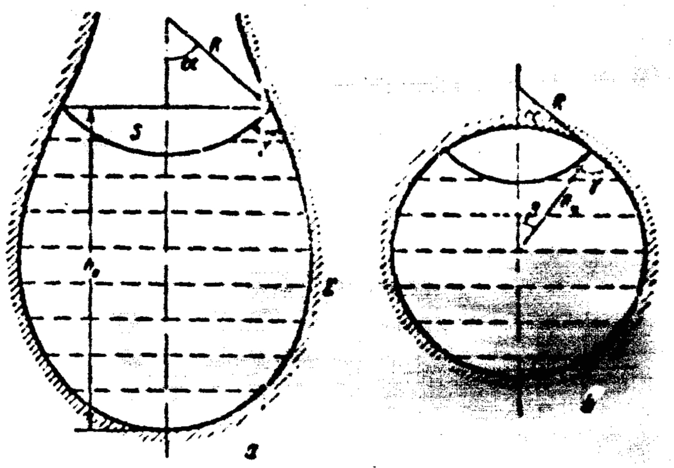

Fig.1

Let us assume that the equation of the surface Σ has the form of $r = F(h)$, where r is the distance of some point from the axis of symmetry, and h is the height from the bottom of the vessel. The problem reduces to finding two parameters: the radius of curvature R of the free surface and the height h_0 of the free surface at the wall. We can determine these with the aid of two equations: /1073

$$\pi\int_0^{h_0} F^2(\xi)\,d\xi - \frac{1}{3}\pi R^3(1-\cos\alpha)^2(2+\cos\alpha) = V,$$

$$F'(h_0) = \cot(\alpha+\gamma) \qquad (\alpha = \mathrm{Arcsin}\,\{F(h_0)/R\}). \tag{1.9}$$

The first equation expresses the constancy of the fluid volume and the second, the constancy of the contact angle.

In the event that the surface Σ is a sphere of radius R (cf. Fig.1b), eqs.(1.9) reduce to

$$\frac{1}{3}\pi R_0^3[4-(1-\cos\beta)^2(2+\cos\beta)]-$$
$$-\frac{1}{3}\pi R^3(1-\cos\alpha)^2(2+\cos\alpha)=V_0,$$
$$h_0=R_0(1+\cos\beta),\qquad R_0\sin\beta=R\sin\alpha,\qquad \gamma+\alpha+\beta=\pi.$$

3. It may happen that the free surface S does not intersect the vessel surface (the bubble being located entirely within the fluid). In this case, the problem of statics is applicable only if mass forces are absent, since the position of the bubble is unstable with respect to these forces: the application of even negligibly small mass forces causes the bubble to move as an integral whole.

If mass forces are absent, the surface S will be a sphere of radius R. Let us place the origin of the spherical coordinate system r, θ, ψ at the center of this sphere. Then, let us consider a surface

$$\xi=\xi(\theta,\psi)=r(\theta,\psi)-R,$$

close to the sphere S. Its double mean curvature $K(\xi)$ can then be presented as (Bibl.1)

$$K(\xi)=\frac{2}{R}+L\xi+O(|\xi|^2), \tag{1.10}$$

where /1074

$$L=-\frac{1}{R^2}\left[\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\psi^2}+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+2\right]. \tag{1.11}$$

We will next discuss certain properties of the operator L. To this end, let us consider the equation

$$L\xi=0. \tag{1.12}$$

A simple check test readily demonstrates that eq.(1.12) has the following non-trivial solutions:

$$\begin{aligned}\xi_1&=cP_1(\cos\theta)=c\cos\theta,\\ \xi_2&=cP_1^1(\cos\theta)\cos\psi=c\sin\theta\cos\psi,\\ \xi_3&=cP_1^1(\cos\theta)\sin\psi=c\sin\theta\sin\psi.\end{aligned} \tag{1.13}$$

where $P_l(x)$ denote Legendre polynomials and $P_l^j(x)$, associated Legendre functions of the first kind. The theory of Legendre polynomials states that eq.(1.12) has no other single-valued and bounded solutions. In fact, any solution of eq.(1.12) that is periodic with respect to ψ may be represented by the following linear combination of functions:

$$\Phi_1^n \sin n\psi, \qquad \Phi_1^n \cos m\psi, \qquad n = 0, 1, 2, \ldots,$$

where $\Phi_1^n(\theta)$ satisfies the equation

$$\frac{1}{\sin\theta}\,\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\Phi_1^n\right)+\left(2-\frac{n^2}{\sin^2\theta}\right)\Phi_1^n = 0.$$

At n = 0, this equation is a particular case of Legendre's equation. Its linearly independent solutions will be P_1 and Q_1 where P_1 is a Legendre polynomial and Q_1 is a Legendre function of the first kind (the function Q_1 is not bounded). At $n \neq 0$, this equation determines the associated Legendre functions P_1^n and Q_1^n. From the theory of these functions, it is known (Bibl.2) that $P_m^n(\theta)$ and $Q_m^n(\theta)$ have a special feature: the function Q_m^n, at no matter what ratio of n to m; and the function P_m^n, on condition that $n > m$.

Hence, the equation

$$L\xi = f \tag{1.14}$$

has solutions only in the functional space f that is orthogonal to the functions ξ_1, ξ_2, and ξ_3. The solutions (1.13) have a simple physical meaning: They describe infinitesimal movements of the sphere S as an integral whole along the axes x, y, and z. In particular, the solution ξ_1 describes the movement of the bubble along the z-axis to the extent c, with an accuracy to within first-order smallness.

Next, let us assume that the fluid is subjected to a uniform field U = az, where a is the field strength, which we consider low (of the order of ξ). The linearized equation (1.8) can then be rewritten in dimensionless form

$$\varepsilon \cos \theta - L\xi = 0 \qquad (\varepsilon = aR^2/\sigma). \tag{1.15}$$

Equation (1.15) is of the type of eq.(1.14) such that the function $f = \xi_1$. Thus, no matter how low the intensity of the mass forces might be, eq.(1.15) has no /1075 bounded solutions. This means that the bubble cannot be in an equilibrium inside the fluid.

4. Let us assume that the static problem is solved, i.e., that the form of the free surface has been determined in one way or another. The free surface, in the equilibrium position, will be denoted by S_0. Let there be motion in the neighborhood of this position and let us correspondingly linearize the problem. Then, eq.(1.1) can be rewritten in the form of

$$\frac{\partial \mathbf{v}}{\partial t} = \nabla\left(\frac{U}{\rho} - \frac{p'}{\rho}\right). \tag{1.16}$$

where $p' = p - p_0(x, y, z)$ – added to p_0 – is the pressure under equilibrium conditions.

The equation of the free surface will become

$$\Phi_s = \Phi_0(x, y, z) + N_1\psi(x, y, z, t) = 0,$$

where ψ is considered to be of first-order smallness. We denote, by N_1, the normalizing factor

$$N_1 = \left[\left(\frac{\partial\Phi_0}{\partial x}\right)^2 + \left(\frac{\partial\Phi_0}{\partial y}\right)^2 + \left(\frac{\partial\Phi_0}{\partial z}\right)^2\right]^{1/2}$$

of the function Φ_0 calculated for points of the surface S_0. Substituting the expression Φ_s in eq.(1.7), we have

$$\frac{\partial\psi}{\partial t} + v_n = 0. \tag{1.17}$$

The condition (1.17) must be satisfied along S_0.

Since, in view of the incompressibility

$$\int_S v_n\, ds = 0, \quad \text{then} \quad \int_S \psi\, ds = \text{const.}$$

It is readily shown that this constant is zero. In the neighborhood of the surface S_0, we introduce the curvilinear coordinate system α, β, δ:

$$x = X(\alpha, \beta) + \frac{\delta}{N_1}\left(\frac{\partial \Phi_0}{\partial x}\right)_{s_0},$$

$$y = Y(\alpha, \beta) + \frac{\delta}{N_1}\left(\frac{\partial \Phi_0}{\partial y}\right)_{s_0}, \quad z = Z(\alpha, \beta) + \frac{\delta}{N_1}\left(\frac{\partial \Phi_0}{\partial z}\right)_{s_0}.$$

The equation of the surface S_0 will be $\delta = 0$. Utilizing the curvilinear coordinates and the close spacing of the surfaces S_0 and S, we rewrite, in linearized form, the equation of the free surface

$$\Phi_s = \frac{\delta}{N_1}\left[\left(\frac{\partial \Phi_0}{\partial x}\right)^2 + \left(\frac{\partial \Phi_0}{\partial y}\right)^2 + \left(\frac{\partial \Phi_0}{\partial z}\right)^2\right] + N_1\psi(\alpha, \beta, t) = 0.$$

Here, $\psi(\alpha, \beta, t) = \psi(X(\alpha, \beta), Y(\alpha, \beta), Z(\alpha, \beta), t)$. It follows that $\delta =$ $= -\psi(\alpha, \beta, t)$. Since $\int\limits_S \delta ds = 0$ at any instant, we have

$$\int\limits_{S_0} \psi\, ds = 0. \tag{1.18}$$

In linearized form, the condition of thermodynamic equilibrium, satisfied along S_0, will be /1076

$$p' = \alpha\Phi_s. \tag{1.19}$$

The boundary-value condition (1.5), in linear formulation, reads

$$\frac{\partial \psi}{\partial \alpha}\sin\gamma + \psi(k_s\cos\gamma - k_\Sigma) = 0 \Big| \tag{1.20}$$

Here, the derivative is taken along the normal to Γ, lying in the plane of contact with S_0; k_s and k_Σ are the curvatures of the sections of the surfaces S_0 and Σ with the plane normal to Γ.

Thus, in its linear formulation, the problem of oscillations of the fluid reduces to determining the functions $v(x, y, z, t)$, $p'(x, y, z, t)$, $\psi(x, y, z, t)$ satisfying eqs.(1.2), (1.16), (1.17), (1.19) and the conditions (1.3) and (1.20).

Note. We will derive the condition (1.20) along the contour Γ, which is satisfied by the function ψ because of eq.(1.5). It is simplest to derive this condition geometrically.

Let us assume that A is a point on the contour Γ along which the wall Σ of the vessel is intersected by the static free surface S_0; n_Σ is the internal

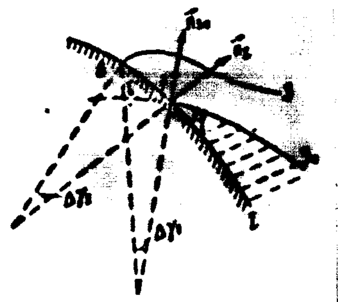

Fig.2

normal to Σ at the point A; n_{s_0} is the external normal to S_0 at the same point (Fig.2). Let us lay the plane G across the vectors $\mathbf{n}_\Sigma$, $\mathbf{n}_{s_0}$. Without restriction of generality, we can assume that the line of intersection of the plane G with the surface S_0 coincides near the point A with the line θ = const. Then, the coordinate α is the length along the curve on which S_0 and G intersect, i.e., along the normal to Γ on the surface S_0. At the point A we have, by definition, $\mathbf{n}_\Sigma \mathbf{n}_{s_0} = \cos\gamma$.

Let us assume that B is the point of intersection of the dynamic free surface S, the vessel wall Σ, and the plane G. Obviously, ψ will then be the distance from the dynamic to the static free surfaces, reckoned from the normal to the latter. Then, with an accuracy to within higher-order smallness, AB = $= -\psi/\sin\gamma$. We will next calculate the variation in the angle γ between the normals $\mathbf{n}_\Sigma \mathbf{n}_s$ on transition from the point A to the point B. This variation $\Delta\gamma$ consists of three terms.

First (Fig.2), the static free surface rotates through the angle $\Delta\gamma_1$ =

$= AC k_s$, where k_s is the curvature of the line of intersection of the surfaces S_0 and G. Obviously, $AC = AB \cos \gamma$. Second, the surface of the walls rotates through the angle $\Delta\gamma_2 = AB k_\Sigma$, where k_Σ is the curvature of the line of intersection of the surfaces Σ and G. Third, in the plane G, the dynamic free surface, together with the static surface, makes the angle $\Delta\gamma_3 = \partial\psi/\partial\alpha$. Taking into account the signs of the angles and the value of AB, we have

$$\Delta\gamma = \Delta\gamma_1 - \Delta\gamma_2 + \Delta\gamma_3 = -(\partial\psi/\partial\alpha) - \psi(k_s \cos\gamma - k_\Sigma)/\sin\gamma.$$

On the other hand, the variation $\Delta\gamma$ is

$$\Delta\gamma = (\partial\gamma/\partial\sigma)AB = -(\partial\gamma/\partial\sigma)(\psi/\sin\gamma),$$

where σ is the length of the path along the line of intersection of Σ and G. Thus, in the general case, the condition over the contour Γ will be

$$\sin\gamma(\partial\psi/\partial\alpha) + \psi(k_s \cos\gamma - k_\Sigma - \text{[illegible]}$$

If the properties of the wall material are identical throughout, then $\partial\gamma/\partial\sigma = 0$ and we arrive at the condition (1.20).

5. This problem can be simplified: Within the scope of the linear theory, only potential flow is in question. To prove this statement, consider the set E of solenoidal vectors $\mathbf{v}$ prescribed in τ – the region bounded by the surfaces Σ and S_0. /1077

The field $\mathbf{v}$ can be represented by superposition

$$\mathbf{v} = \mathbf{u} + \mathbf{w} \tag{1.21}$$

of the potential field $\mathbf{u} = \nabla\varphi$ and the vortical $\mathbf{w}$ field, where φ is a function harmonic in τ and $\mathbf{w}$ satisfies the condition $\nabla\mathbf{w} = 0$.

Equation (1.21) can be realized by innumerable methods. This raises the question of the most complete isolation of the potential component.

We will conditionally consider the subdivision complete if the vector $\mathbf{w}$ is, in one way or another, orthogonal to the vector $\mathbf{u}$. In order to make this definition more precise, a metric must also be introduced into the set E, thus transforming this set into a space. It appears most logical to utilize the energy metric

$$(\mathbf{v}_1, \mathbf{v}_2)_E = \int_\tau \mathbf{v}_1 \mathbf{v}_2 d\tau.$$

Thus, in this case, completeness of subdivisions means that $(\mathbf{u}, \mathbf{w})_E = 0$.

We will denote by E_u the subspace of the potential vectors and determine the set E_w of the vectors $\mathbf{w} \in E$ whose normal component is zero over $\Sigma + S$. We will prove that the set E_w belongs to the orthogonal complement of E_u. To this end, we will utilize Green's formula, the condition that the field be solenoidal, and the condition $w_n = 0$ on Σ and S_0:

$$(\mathbf{u}, \mathbf{w})_E = \int_\tau \nabla\varphi \mathbf{w}\, d\tau = \int_{\Sigma+S_0} \varphi w_n\, ds = 0.$$

This provides the means for the structural isolation of the potential field component. First, we consider the following Neumann problem

$$\Delta\varphi = 0 \text{ в } \tau, \qquad (\partial\varphi/\partial n) = (\mathbf{v}\mathbf{n}) \text{ at } \Sigma \text{ and } S_0. \tag{1.22}$$

The solution of the problem (1.22) makes every vector field $\mathbf{v}$ correspond to the gradient field $\mathbf{u} = \nabla\varphi$. We will write this fact as: $\mathbf{u} = \Pi_u \mathbf{v}$, where Π_u is the operator of orthogonal mapping of E onto E_u. The vector $\mathbf{w}$ is determined by the formula $\mathbf{w} = \mathbf{v} - \mathbf{u}$ or $\mathbf{w} = \Pi_u \mathbf{v}$.

On having determined the mapping operators, we return to our problem. Applying the operators Π_u and Π_w to both sides of eq.(1.16)

$$\Pi_u\left(\frac{\partial \mathbf{v}}{\partial t} + \nabla\frac{p'}{\rho} - \nabla\frac{U}{\rho}\right) = 0, \quad \Pi_w\left(\frac{\partial \mathbf{v}}{\partial t} + \nabla\frac{p'}{\rho} - \nabla\frac{U}{\rho}\right) = 0,$$

we have (because of the linearity of the operation of orthogonal mapping)

$$\frac{\partial \nabla\varphi}{\partial t} = \nabla\left(\frac{U}{\rho} - \frac{p'}{\rho}\right), \tag{1.23}$$

/1078

$$\frac{\partial \mathbf{w}}{\partial t} = 0. \tag{1.24}$$

Equation (1.23) yields the Cauchy-Lagrange integral

$$\frac{\partial \varphi}{\partial t} = \frac{U}{\rho} - \frac{p'}{\rho}. \tag{1.25}$$

Thus, the pressure field is determined solely by the potential component of the velocity field.

Note. It is exactly this fact that warrants introducing the energy metric and the corresponding expression for the potential component.

It follows from eq.(1.25) that, from the viewpoint of the linear field theory, $\mathbf{w}$ does not change in time, meaning that the point vorticity is constant.

Using the definition of the potential component (1.22) we will rewrite eq.(1.17):

$$\frac{\partial \psi}{\partial t} + \frac{\partial \varphi}{\partial n} = 0. \tag{1.26}$$

The condition (1.26) shows that, in the theory developed here, the free surface is determined uniquely by the potential component.

The pressure distribution along the free surface also is independent of $\mathbf{w}$, so that eq.(1.19) can be written in the form of

$$\rho \frac{\partial \varphi}{\partial t} + \sigma L \psi = U(\psi). \tag{1.27}$$

Thus, the problem reduces to determining the function φ harmonic in τ and the function ψ according to eqs.(1.3), (1.18), (1.20), (1.26), and (1.27).

This proves that the assumption of potentiality, usually adopted in linear problems, is a logical consequence of exactly this linearity.

Section 2. Solvability of Problems of the Theory of Linear Oscillations and Structure of the Spectrum

1. The problem formulated at the end of the preceding Section can be reduced to a one-operator equation. Let us use the Neumann operator H. The expression

$$\varphi(P) = Hf(Q), \quad Q \in S_0, \quad P \in \tau,$$

is to mean that the function f(Q) [provided this function satisfies the condition (1.18)] corresponds to a function harmonic in τ, whose normal derivative becomes zero on Σ and f(Q) on S_0. The operator H is positive-integral with a weak singularity (Bibl.3). Hence, it is completely continuous. Its self-conjugateness derives from the symmetry of Green's function of the Neumann problem.

The condition (1.26) makes it possible to write

$$\varphi = -H\frac{\partial\psi}{\partial t}. \tag{2.1}$$

Using eq.(2.1) and considering that $U(\psi) = -\rho k\psi$ for small ψ, we rewrite eq.(1.27): /1079

$$-H\frac{\partial^2\psi}{\partial t^2} + \frac{\sigma}{\rho}L\psi = -k\psi. \tag{2.2}$$

Equation (2.2) is an integrodifferential equation containing only one unknown function $\psi(\alpha, \beta, t)$ satisfying t $\frac{\partial\psi}{\partial\alpha} + B\psi = 0,$ lition (1.20). This condition, in the variables α, β, has the form

$$\tag{2.3}$$

where B is the prescribed function of the point of the curve Γ where the surfaces S_0 and Σ intersect.

2. A fundamental problem of the theory of oscillations is to find the fundamental oscillation modes. We will pose $\psi = -f \cos \mu t$.

Equation (2.2) has the form

$$\mu^2 Hf = Af \quad (A = -(\sigma/\rho)L - kI), \tag{2.4}$$

where I is a unit operator. The operator L is self-adjoint. This fact is rather obvious, since the operator L describes a conservative system. First, let us consider the particular case where the operator has the form of eq.(1.11) and is prescribed with respect to a set of functions that depend only on the width θ, and where $B = 0$. Then,

$$Lf = -\frac{1}{R^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + 2f\right], \tag{2.5}$$

$$f'(\theta_0) = f'(\theta_1) = 0, \tag{2.6}$$

where θ_0, θ_1 are the values of the angle θ corresponding to the vessel walls. By determining the scalar product

$$(f, g) = \int_{\theta_0}^{\theta_1} \sin\theta\, fg\, d\theta,$$

we can prove the self-adjointness of L:

$$(Lf, g) = -\frac{1}{R^2}\int_{\theta_0}^{\theta_1} \sin\theta\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + 2f\right] g\, d\theta =$$

$$= \frac{1}{R^2}\int_{\theta_0}^{\theta_1}\left(\frac{\partial f}{\partial\theta}\frac{\partial g}{\partial\theta}\sin\theta - 2fg\sin\theta\right)d\theta =$$

$$= -\frac{1}{R^2}\int_{\theta_0}^{\theta_1}\left[f\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial g}{\partial\theta}\right) + 2fg\sin\theta\right]d\theta =$$

$$= -\frac{1}{R^2}\int_{\theta_0}^{\theta_1}\sin\theta\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial g}{\partial\theta}\right) + 2g\right] f\, d\theta = (f, Lg).$$

The proof for the self-adjointness of L in the general case is just as elementary. Its realization requires using Green's formulas instead of an integration by parts. The self-adjointness of L directly implies the self-adjointness of the operator A from eq.(2.4).

We will consider only the case where the operator A is positive-definite. Since $(Af, f) = 2\Pi$, where Π is the potential energy of the mass forces and of /1080 the surface tension forces, our assumption will mean that, when in an equilibrium position, the potential energy of the system has a discrete minimum, i.e., the free surface of the liquid is statically stable.

This leads us to the standard problem of eigenvalues (2.4), where L is a wholly continuous, self-adjoint, positive operator, and A is a self-adjoint positive-definite operator. On the basis of the known theorems of the spectral theory of linear operators (Bibl.4), we can draw the following conclusions:

1) The spectrum of the problem (2.4) is discrete and of finite multiplicity, with a unique limiting point $\mu = \infty$. This means that, under the conditions considered here, there exists an infinite multiplicity of eigenfrequencies μ_n such that $\mu_n \to \infty$ at $n \to \infty$ and each eigenfrequency corresponds to a finite number of the possible modes of fundamental oscillations.

2) The spectrum is entirely located on the real semiaxis, i.e., all fundamental oscillations are stable.

3) The system of eigenfunctions is complete with respect to Friedrichs' norm.

4) All fundamental oscillations and eigenfrequencies can be derived by means of Ritz's combination principle.

Note. The reasoning in this Section was essentially based on conditions of the type of eq.(2.3). If the bubble is entirely within the fluid, this reasoning retains its validity in the space of the functions orthogonal to the functions ξ_1, ξ_2, and ξ_3 from eq.(1.13).

Section 3. Elementary Problems of the Oscillation Theory

1. The findings of the preceding Section indicate that, fundamentally, the problem of small oscillations is not particularly complex but its effective solutions are nearly unknown.

An elementary problem of this kind is the oscillation of a layer of weightless fluid (Bibl.5) bounded by a sphere Σ of radius R_2 and by a free surface, on the assumption that the latter is a sphere of radius R_1, concentric with Σ (Fig.3).

We set $q = R_2/R_1$ and introduce the dimensionless spherical coordinates r, θ, λ taking R_1 as the characteristic scale and placing the origin of the coordinates at the center of the spheres S, Σ. Then the condition of non-leakage will be written in the form

$$\partial\Phi/\partial r = 0 \quad \text{at} \quad r = q. \tag{3.1}$$

The set of harmonic functions satisfying eq.(3.1) will then be

$$\Phi_{n,m}^{1,2} = \mathrm{const}\left(\frac{n+1}{n} q^{-2n-1} r^{n} + r^{-n-1}\right) P_n^m(\cos\theta) \frac{\sin}{\cos}(m\lambda). \tag{3.2}$$

We will seek fundamental oscillations in the form of $\varphi = \Phi(r, \theta, \lambda) \sin \mu t$, $\psi = f\mu^{-1} \cos \mu t$, where Φ is a function of the type of eq.(3.2). Then, the kinematic condition (1.26) and the dynamic condition (1.27) will read

$$f = \partial\Phi/\partial r, \; \rho\mu^2\Phi + \sigma L f = 0 \quad \text{at} \quad r = 1, \tag{3.3}$$

where L is determined by eq.(1.11). Eliminating the function f from eq.(3.3), we arrive at the equation for μ. The final result will be /1081

$$\mu_n^2 = \frac{\sigma}{\rho}(n^2 - 1)(n + 2)\frac{1 - q^{-2n-1}}{1 + [(n+1)/n]q^{-2n-1}}.$$

Note the that oscillation frequencies are independent of m. Hence, all the fundamental oscillations, whose velocity potential is described by the functions $\Phi_{nm}^{1,2}$ ($m = 0, 1, \ldots n$), have the same frequency. Thus, the number μ_n has the

Fig.3

Fig.4

multiplicity 2n + 1. The value of $\mu_1 = 0$ corresponds to nontrivial solutions of the equation $Lf = 0$, which describe the motion of the bubble as a whole. The least nonzero eigenfrequency corresponds to n = 2.

$$\mu_2^2 = 12\frac{\sigma}{\rho}\frac{1-q^{-5}}{1+\frac{3}{2}q^{-5}}, \quad \mu_3^2 = 40\frac{\sigma}{\rho}\frac{1-q^{-7}}{1+\frac{4}{3}q^{-7}}, \quad \mu_4^2 = 90\frac{\sigma}{\rho}\frac{1-q^{-9}}{1+\frac{5}{4}q^{-9}}.$$

2. The example described above exhausts the known exact solutions of this problem. The known approximate solutions also are few in number. Chernous'ko (Bibl.6) gives a general method for the solution of the problem of bubble motion, on condition that the volume of the bubble is small compared with the volume of the fluid. Another possible case might be: The depth of the fluid is small (cf. Fig.4) in comparison with the linear dimensions of its free surface. In this case, the problem can be greatly simplified by utilizing the narrow-band asymptotic methods (Bibl.7).

To simplify the calculations and to better illustrate the procedure for the asymptotic analysis of the problem, let us consider the elementary case of a weightless fluid in an axisymmetric vessel. The free surface represents a sphere when in equilibrium position. We will consider only axisymmetric oscillation modes. The equations for the fluid surface in equilibrium position and for the vessel walls will be written in the form of

$$r = R, \qquad r = R + s(\theta) = R + \varepsilon f(\theta). \tag{3.4}$$

The problem is to find the function $\Phi(t, r, \theta)$ satisfying in τ the Laplace /1082 equation

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial\Phi}{\partial\theta}\right) = 0, \tag{3.5}$$

and the function $\psi(t, \theta)$ from the conditions

$$\partial\Phi/\partial n = 0 \quad \text{at} \quad r = R + s, \tag{3.6}$$

$$\partial\Phi/\partial t + \varepsilon L\psi = 0 \quad \text{at} \quad r = R, \tag{3.7}$$

$$\partial\psi/\partial t = -\partial\Phi/\partial r \quad \text{at} \quad r = R, \tag{3.8}$$

$$\partial\psi/\partial r + B\psi = 0 \quad \text{at} \quad r = R,\ \theta = \theta_0, \tag{3.9}$$

where θ_0 is the value of the angle θ at the point of contact between the unperturbed free surface and the walls. First let us consider the subproblem of determining the function $\Phi(t, r, \theta)$ which satisfies eq.(3.5), the condition (3.6), and the condition

$$\Phi(t, R, \theta) = a(t, \theta), \tag{3.10}$$

where a is a prescribed function of its variables. We then construct the asymptotic solution of this problem for $\varepsilon \to 0$. Let us substitute $r = R + \varepsilon\xi$ into eq.(3.5):

$$\frac{\partial}{\partial\xi}\Big[\text{[illegible]} \tag{3.11}$$

$$\Delta\Phi = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial\Phi}{\partial\theta}\right). \tag{3.12}$$

Employing the notation of eq.(3.12) we can write [see eq.(1.11)]

$$L = -\frac{1}{R^2}(\Delta + 2).$$

The condition (3.6) can be written as

$$\frac{\partial\Phi}{\partial\xi} = \frac{\varepsilon^2}{R^2}\frac{\partial\Phi}{\partial\theta} f'(\theta) + O(\varepsilon^3).$$

We then construct the solution of the subproblem in form of the series

$$\Phi = \Phi_0 + \varepsilon\Phi_1 + \varepsilon^2\Phi_2 + \ldots . \tag{3.13}$$

For Φ_0, we have

$$\frac{\partial^2\Phi_0}{\partial\xi^2} = 0 \text{ in } \tau, \quad \frac{\partial\Phi_0}{\partial\xi} = 0 \text{ at } \xi = l, \quad \Phi_0 = a \text{ at } \xi = 0.$$

Hence, $\Phi_0 = a(t, \theta)$.

By exactly the same reasoning, we obtain $\Phi_1 \equiv 0$. The function Φ_2 satisfies the following problem

$$\begin{aligned} &\frac{\partial^2\Phi_2}{\partial\xi^2} = -\frac{1}{R^2}\Lambda\Phi_0 = -\frac{1}{R^2}\Lambda a(t, \theta) \text{ in } \tau, \\ &\Phi_2 = 0 \quad \text{at} \quad \xi = 0, \quad \frac{\partial\Phi_2}{\partial\xi} = \frac{1}{R^2} f'(\theta)\frac{\partial a}{\partial\theta} \quad \text{at} \quad \xi = l. \end{aligned} \tag{3.14}$$

The solution of the problem (3.14) is given by the formula

$$\Phi_2 = \frac{\xi}{R^2}\left[\left(l - \frac{\xi}{2}\right)\Lambda a + f'(\theta)\frac{\partial a}{\partial\theta}\right].$$

The remaining terms of the series (3.13) are not as easy to calculate. /1083
Generally, the series (3.13) diverges. Moiseyev (Bibl.7) described the a priori conditions for its asymptoticity. We will consider these conditions satisfied and confine ourselves to calculating Φ_0 and Φ_2. Returning to the old variables, we have

$$\Phi = a(t, \theta) + \frac{r-R}{R^2}\left[\left(z(\theta) - \frac{r-R}{2}\right)\Lambda a + z'(\theta)\frac{\partial a}{\partial\theta}\right]. \tag{3.15}$$

With the aid of eq.(3.15) we can rewrite eqs.(3.7) and (3.8)

$$\frac{\partial a}{\partial t} = \frac{g}{R^2}(\Lambda\psi + 2\psi), \quad \frac{\partial\psi}{\partial t} = -\frac{1}{R^2}\left(z\Lambda a + z'\frac{\partial a}{\partial\theta}\right). \tag{3.16}$$

Thus, the problem is reduced to finding two functions that do not depend on the radius vector.

To find the eigenfrequencies, we pose $a = \alpha(\theta)e^{\omega t}$, $\psi = \beta(\theta)e^{\omega t}$ after which it follows from eq.(3.16) that

$$[illegible]\frac{\sigma}{R^2}(A+2)\beta, \qquad \omega\beta = -\frac{1}{R^2}(zA\alpha + z'\alpha').$$

This problem is equivalent to the following eigenvalue problem:

$$\sigma R^{-4}(A+2)(zA\beta + z'\beta') = -\omega^2\beta.$$

The latter problem is fundamentally simpler than the initial problem, since the sought function β depends only on the variable θ.

Section 4. Case of Low Surface Tension (Basic Equations and Static Problem)

1. The problem of small oscillations of an ideal fluid in the presence of surface tension, formulated in Section 1, can be effectively solved whenever the surface tension is low compared with the mass forces. Proceeding from this premise, the equilibrium and the natural oscillations of a heavy ideal fluid (standing waves) in a vessel will be examined below. Because of the above statements, we considered only potential flow in Section 1.

The velocity potential of $\varphi(x, y, z, t)$ in the flow region satisfies the Laplace equation and, at the walls of the vessel Σ, the condition of non-leakage

$$\Delta\varphi = 0, \quad \partial\varphi/\partial N = 0 \text{ at } \Sigma. \tag{4.1}$$

where N is the internal normal to the vessel wall. Let us assume that the z-axis is directed vertically upward and the plane $z = 0$ coincides with the surface of the fluid at rest, in the absence of surface tension ($\sigma = 0$). At the free surface $z = \zeta(x, y, t)$, the kinematic and dynamic conditions (g = acceleration of gravity)

$$\frac{\partial\zeta}{\partial t} + \frac{\partial\zeta}{\partial x}\frac{\partial\varphi}{\partial x} + \frac{\partial\zeta}{\partial y}\frac{\partial\varphi}{\partial y} - \frac{\partial\varphi}{\partial z} = 0,$$
$$\frac{1}{g}\frac{\partial\varphi}{\partial t} + \frac{(\nabla\varphi)^2}{2g} + \zeta + h_0 + \frac{\sigma}{\rho g}K(\zeta) = 0 \quad \text{at} \quad z = \zeta. \tag{4.2}$$

are satisfied. The second condition in eq.(4.2) is a consequence of the Cauchy-

Lagrange integral and of eq.(1.4), while h_0 is a constant.

The double mean surface curvature has the form /1084

$$K(\zeta) = \mp \frac{\zeta_{xx}(1+\zeta_y^2) - 2\zeta_x\zeta_y\zeta_{xy} + \zeta_{yy}(1+\zeta_x^2)}{(1+\zeta_x^2+\zeta_y^2)^{3/2}}. \qquad (4.3)$$

where the subscripts x, y denote partial derivatives; the upper sign must be used in the event that the fluid is located below the surface $z = \zeta$, and the lower sign, if the fluid is located above the surface $z = \zeta$. The function $\zeta(x, y, t)$ may be ambiguous and its different branches correspond to different signs in eq.(4.3).

The constant h_0 will be so determined that, in the equilibrium case ($\nabla\varphi = 0$) the potential can be taken as independent of t. Prescribing the static form of the free surface in the form of

$$z = \zeta = -h_0 + h(x, y). \qquad (4.4)$$

we obtain from eq.(4.2) the equation for h:

$$\rho g h + \sigma K(h) = 0. \qquad (4.5)$$

The boundary condition for eq.(4.5) is the prescribed contact angle for the contour Γ_1, where the free surface osculates the walls. After having determined h(x, y), it is possible to find h_0 if it is assumed that the fluid volume is the same no matter whether $\sigma = 0$ or $\sigma \neq 0$.

In the dynamic problem, the free surface will be given as

$$z = -h_0 + h(x, y) + f(x, y, t). \qquad (4.6)$$

The oscillation amplitude is considered small and eqs.(4.2), (4.3) are linearized with respect to the functions φ, f, while simultaneously referring them to the static free surface [eq.(4.4)]:

$$f_t + h_x\varphi_x + h_y\varphi_y - \varphi_z = 0, \quad \frac{1}{g}\varphi_t + f + \frac{\sigma}{\rho g}K_1(f) = 0 \quad \text{at } z = -h_0 + h. \qquad (4.7)$$

where $K_1(f)$ is the linear - with respect to f - part of the increment $K(h + f) - K(h)$, which, on using eq.(4.3), we can write in the form

$$K_1(f) = \mp (1 + h_x^2 + h_y^2)^{-3/2} [f_{xx}(1 + h_y^2) + 2(h_y h_{xx} - h_x h_{xy}) f_y + 2(h_x h_{yy} - h_y h_{xy}) f_x - 2 h_x h_y f_{xy} + f_{yy}(1 + h_x^2)] - 3K(h)(h_x f_x + h_y f_y)(1 + h_x^2 + h_y^2)^{-1}. \tag{4.8}$$

In addition, the function f satisfies the homogeneous boundary condition of the type of eq.(1.20) for the contour Γ_1. The resulting boundary-value problem for the functions φ, f is linear and homogeneous; however, prior to its solution, it is necessary to find the function $h(x, y)$ satisfying the nonlinear equation (4.5).

2. We introduce the dimensionless parameter

$$\varepsilon = \sqrt{\sigma} / (l \sqrt{\rho g}), \tag{4.9}$$

where l is the characteristic linear dimension of the vessel, which we will consider small. Then eq.(4.5) will become

$$h + \varepsilon^2 K(h) = 0$$

and will contain a small parameter with higher-order derivatives. The boundary condition for this equation (angle of contact between wall and free surface) is given for an a priori unknown (generally speaking) three-dimensional contour Γ_1. If $\sigma = \varepsilon = 0$, the solution is $h \equiv 0$ and the contour Γ_1 changes into the planar contour Γ over which the plane $z = 0$ intersects the vessel walls. /1085

We make the logical inference that the contour Γ_1 is close to Γ when $0 < \varepsilon \ll 1$, while the solution of eq.(4.5) significantly differs from zero only in the narrow region near the contour Γ_1. The justification for this hypothesis is the fact that the approximate solution obtained below exhibits such properties. To determine the solution, we will make use of the boundary-layer method (Bibl.8, 9).

Into the xy-plane, let us introduce the following curvilinear orthogonal coordinates: n, the distance along the normal from a given point M to the contour Γ; s, the arc length of the contour Γ from the point taken as the origin of coordinates, along the normal going through the point M (Fig.5). Note that

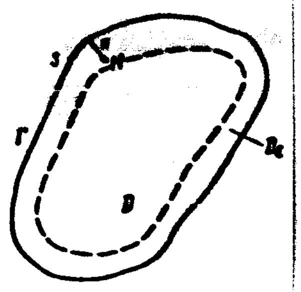

Fig.5

specifying the free surface in the form of the function h(n, s) displays two shortcomings: first, the coordinates n, s are determined randomly for the points separated from the contour Γ by distances of the order of its radius of curvature; second, the function h(n, s), as was pointed out above, may be ambiguous. These shortcomings can be eliminated if the free surface is specified in parametric form. We will, however, employ the coordinates n, s since this will simplify the calculations without affecting the final results. We will only assume that the contour Γ has no vertex points and that the radius of its curvature everywhere greatly exceeds ϵ. Then, in the narrow region D_ϵ of the plane xy adjoining the contour Γ and having the width $\sim \epsilon$, the coordinates n, s are unambiguously defined.

Next, we perform the transformation of the prolongation of $h = \epsilon H$, $n = \epsilon u$, $s = s$. At the boundary contour, because of the finiteness of the angle of contact, we have $h_n \simeq H_u \sim 1$. Moreover, in the region D_ϵ we clearly have $u \sim s \sim \sim \ell$. We will assume that the function H(u, s) and all of its derivatives are finite [of the order of $O(1)$] in D_ε. Then we obtain, in D_ε, the estimates

$$h \sim \varepsilon l, \quad h_n \sim 1, \quad h_s \sim \varepsilon, \quad h_{nn} \sim (l\varepsilon)^{-1}, \quad h_{ns} \sim l^{-1}, \quad h_{ss} \sim \varepsilon l^{-1}. \tag{4.10}$$

Let us now pass to the variables n, s in eq.(4.5); taking into account eqs.(4.9), (4.10), and the evident equality $(\partial n/\partial x)^2 + (\partial n/\partial y)^2 = 1$, we arrive at the equation

$$h = \pm \frac{l^2\varepsilon^2 h_{nn}}{(1+h_n^2)^{3/2}}, \tag{4.11}$$

which is valid in D_ε with an accuracy to within higher orders of smallness. Here the sign is selected on the basis of the same reasoning as for eq.(4.3). Derivatives with respect to s do not enter in eq.(4.11) so that this relation may be regarded as an ordinary differential equation. It has the same form as the equation of the free surface for the case of a plane wall (Bibl.1).

Let us write the first integral of eq.(4.11): /1086

$$\frac{h^2}{2} = \mp \frac{l^2\varepsilon^2}{(1+h_n^2)^{1/2}} + C_1. \tag{4.12}$$

Clearly, $h_n \to 0$, $h \ll \varepsilon$ at a distance from the contour Γ, and in eq.(4.12) the upper sign must be taken, since the liquid lies below the free surface. Hence, with an accuracy to higher orders of smallness, we have $C_1 = l^2\varepsilon^2$. At the points where the free surface is vertical the sign changes in eq.(4.12) and we have $h_n = \infty$, $|h| = h_* = l\varepsilon\sqrt{2}$.

We then solve eq.(4.12) with respect to h_n, taking into account the value of C_1:

$$h_n = \pm \frac{h\sqrt{4l^2\varepsilon^2 - h^2}}{h^2 - 2l^2\varepsilon^2}.$$

Figure 6 illustrates the elementary forms of the free surface at the wall, for various angles of inclination of the walls and various angles of contact. It is evident that the signs of h and h_n are opposite at $|h| < h_*$ but identical at $|h| > h_*$. This predetermines the selection of the plus sign in the preceding formula, in every case,

$$h_n = -\frac{h\sqrt{4l^2\varepsilon^2 - h^2}}{h^2 - 2l^2\varepsilon^2}. \qquad (4.13)$$

Note that it follows from eq.(4.13) that $|h| \leq 2l\varepsilon$. Integrating eq.(4.13), we have

$$n = C - \sqrt{4l^2\varepsilon^2 - h^2} - \frac{l\varepsilon}{2}\ln\left[\frac{1-\sqrt{1-(h/2l\varepsilon)^2}}{1+\sqrt{1-(h/2l\varepsilon)^2}}\right]. \qquad (4.14)$$

Figure 7 shows the intersection of the free surface [eq.(4.4)] with the vertical plane z, n. Here, n is the internal normal to the planar contour Γ in

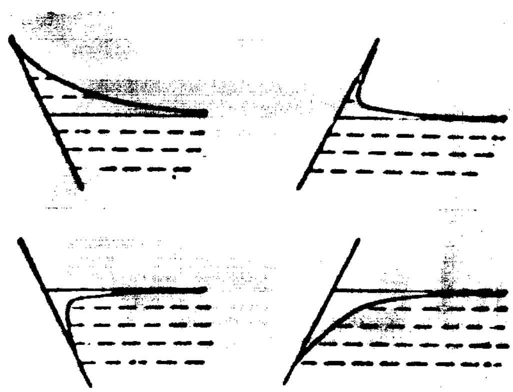

Fig.6

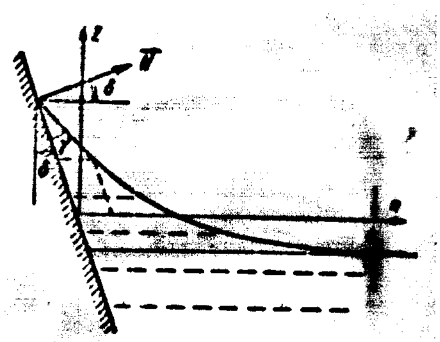

Fig.7

the plane z = 0, N is the internal normal to the vessel wall, and the included angle δ is the angle of inclination of the walls to the vertical, and $\zeta \to -h_0$ at $n \to \infty$. If the contact angle γ is prescribed, we have the following condition at the point P of contact of the free boundary with the wall

$$h_n = -\cot(\gamma + \delta). \qquad (4.15)$$

A simple geometric examination of Figs.6 and 7 shows that, in all cases, at /108 the point P we have sgn h = sgn cos $(\gamma + \delta)$, while sgn sin $(\gamma + \delta)$ is opposite to the sign which must be selected in eq.(4.12). Considering this and substituting eq.(4.15) into eq.(4.12), we find the value of h at the wall (at point P):

$$h_1 = l\varepsilon\sqrt{2}\,\frac{\cos(\gamma+\delta)}{\sqrt{1+\sin(\gamma+\delta)}}. \qquad (4.16)$$

On the other hand, the points on the wall satisfy, in the neighborhood of the point P, the equation $n = -z \tan \delta$, or

$$n = n_1 = (h_0 - h_1) \tan \delta. \tag{4.17}$$

Substituting eqs.(4.16), (4.17) into eq.(4.14), we find the constant

$$C = \left[h_0 - \frac{l\varepsilon\sqrt{2}\cos(\gamma+\delta)}{\sqrt{1+\sin(\gamma+\delta)}}\right]\tan\delta + l\varepsilon\sqrt{2}[1+\sin(\text{[illegible]}) +$$
$$+ \frac{l\varepsilon}{2}\ln\left[\frac{1-\sqrt{[1+\sin(\gamma+\delta)]/2}}{1+\sqrt{[1+\sin(\gamma+\delta)]/2}}\right].$$

Let us now estimate the function h and its derivatives at a certain distance from the walls. At $n \sim \ell$, eq.(4.14) will yield

$$-l\varepsilon \ln\left[1 - \sqrt{1-(h/2l\varepsilon)^2}\right] = O(l).$$

whence, on resolving the radical for $h \ll \ell\varepsilon$, we have

$$h \sim l\varepsilon e^{-k/\varepsilon}, \quad k = O(1) > 0.$$

Similar estimates are valid for the derivatives at finite distances from the wall:

$$h_x \sim h_y \sim e^{-k/\varepsilon}, \quad h_{xx} \sim h_{xy} \sim h_{yy} \sim e^{-k/\varepsilon}(l\varepsilon)^{-1}.$$

This conclusion is not completely rigorous, since we used an approximate solution which is valid only in the region D_ε. However, outside D_ε, the derivatives h_x, h_y are small and eq.(4.5) can be linearized, selecting the minus sign in eq.(4.3) (the liquid at a distance from the walls is located below the free surface). This will yield the equation $h = \ell^2\varepsilon^2\Delta h$ whose solution, within the region, may not have positive maxima or negative minima (Bibl.10). Therefore, if the above estimates hold for a certain contour, they will also hold everywhere within that contour. Accordingly, we may assume that $h \equiv 0$ outside the region D_ε, with an accuracy to within an error smaller than any degree of ε.

Let us next determine h_0 from the condition $v = h_0 D$, expressing the equality of fluid volumes at $\sigma = 0$ and $\sigma \neq 0$ (v is the fluid volume between the surface $z = \zeta$ and the plane $z = -h_0$, D is the area of the region xy bounded by the contour Γ in Fig.5). Let us assume that S is the area in the plane zn bounded by the wall, by the free surface, and by the line $z = -h_0$ (Fig.7). Then, obviously,

$$v = \oint_\Gamma S(s)ds.$$

In calculating S, it is more convenient to perform the integration with respect to z, since h(n) may be an ambiguous function, whereas the function n(h) from eq.(4.14) is unambiguous: /1088

$$S = \int_{-h_0}^{h_1-h_0} [n(h) - (-z\tan\delta)]dz = \int_0^{h_1} n(h)dh + \frac{1}{2}\tan\delta(h_1^2 - 2h_1h_0) =$$
$$= n_1h_1 - \int_0^{h_1} \frac{h}{h_n}dh + \frac{1}{2}\tan\delta(h_1^2 - 2h_1h_0).$$

Here, integration by parts was performed. Hereafter, we will use eqs.(4.17), (4.13), and (4.16):

$$S = -\frac{1}{2}\tan\delta\, h_1^2 - \int_0^{h_1} \frac{h^2 - 2l^2\varepsilon^2}{\sqrt{4l^2\varepsilon^2 - h^2}}dh = -\frac{1}{2}\tan\delta\, h_1^2 + \frac{1}{2}h_1\sqrt{4l^2\varepsilon^2 - h_1^2} =$$
$$= l^2\varepsilon^2\{-\tan\delta[1 + \sin(\gamma+\delta)] + \cos(\gamma+\delta)\} = l^2\varepsilon^2(\cos\gamma - \sin\delta)/\cos\delta. \quad (4.18)$$

Substituting in the equality $v = h_0D$ the formulas for v and S, we have

$$h_0 = \frac{l^2\varepsilon^2}{D}\oint_\Gamma \frac{\cos\gamma(s) - \sin\delta(s)}{\cos\delta(s)}ds. \quad (4.19)$$

Equation (4.19) takes into account the dependence of the contact angle and of the wall inclination on the points of the wall.

We will then transform eq.(4.8), passing to the variables n, s, utilizing eq.(4.5) and the estimates (4.10), and discarding the values of higher-order smallness with respect to ϵ:

$$K_1(f) = (1 + h_n^2)^{-3/2}[\Delta f + h_n^2(f_{xx}n_y^2 - 2f_{xy}n_xn_y +$$
$$+ f_{yy}n_x^2)] + 3(1 + h_n^2)^{-1/2}h_n(f_xn_x + f_yn_y)l^{-2}\varepsilon^{-2}. \quad (4.20)$$

where Δ is the Laplace operator in the xy-plane. The partial derivatives n_x, n_y are simply the direction cosines of the internal normal n to the contour Γ, plotted from the point $M(x, y)$ (Fig.5) with the coordinate axes. Outside the region D_ε, eq.(4.20) yields – taking the selection of the sign into account – $K_1 = -\Delta f$.

Section 5. Natural Oscillations in the Presence of Small Surface Tension

1. Let us solve the problem of the natural oscillations of a fluid, given the following additional assumptions: 1) the vessel walls are vertical in the neighborhood of the free surface, i.e., $\mathbf{N} = \mathbf{n}$, $\delta = 0$ in the neighborhood of the contour Γ; here h is an unambiguous function x, y and the fluid everywhere is located below the free surface; 2) the contact angle is constant: $\gamma = \text{const}$; 3) $\partial h/\partial s = 0$, i.e., $h = h(n)$ everywhere in the neighborhood of the contour Γ; then Γ_1 is a planar contour and its projection onto the xy-plane coincides with Γ. This last assumption is correct if, for example: a) the vessel walls in the neighborhood of Γ form a circular cylinder (then the vessel need not even be axisymmetric), b) the vessel walls in the neighborhood of Γ are plane, c) $\gamma \equiv \pi/2$ for an arbitrary vessel.

These assumptions greatly simplify the condition (1.20) for the contour Γ. In fact, for the points on the dynamic free surface we have /1089

$$\frac{(\mathbf{k} - \nabla h - \nabla f, \mathbf{n})}{\sqrt{1 + (\nabla h + \nabla f)^2}} = \cos\gamma \quad \text{at} \quad (x, y) \in \Gamma,$$

where k is the unit vector of the z-axis, and the gradient is taken with respect to the variables x, y. Considering that $h = h(n)$, we can write

$$\frac{-h_n - f_n}{\sqrt{1 + (h_n + f_n)^2 + f_s^2}} = \cos\gamma \quad \text{along } \Gamma.$$

On the other hand, $-h_n(1 + h_n^2)^{-1/2} = \cos\gamma$ by virtue of the conditions for the static free surface. Therefore, ultimately in the linear approximation with respect to f, we will obtain the condition $\partial f/\partial n = 0$ at Γ. This condition is a simplified variant of the condition (1.20), under the above assumptions.

Passing now to the solution of the problem of natural oscillations, we pose

$$\varphi = e^{i\omega t}\Phi(x, y, z), \quad f = e^{i\omega t}F(x, y),$$

where ω is the sought eigenfrequency. The function Φ is harmonic in the region occupied by the fluid, and satisfies the condition $\partial\Phi/\partial N = 0$ at the wall. Instead of eqs.(4.7) we then have, at $z = h - h_0$,

$$i\omega F + h_x\Phi_x + h_y\Phi_y - \Phi_z = 0, \tag{5.1}$$

$$i\omega g^{-1}\Phi + F + P\epsilon^2 K_1(F) = 0. \tag{5.2}$$

Moreover, we have the condition $\partial F/\partial n = 0$ along Γ. The problem is to determine the eigenfrequencies ω at which the linear homogeneous boundary-value problem for Φ, F has a nonzero solution, and to find the functions Φ, F themselves in this case (eigenfunctions).

In the absence of surface tension ($\epsilon = 0$, $h = h_0 = 0$) the properties of the problem of natural oscillations are well-known (Bibl.3). In this case, we have a discrete spectrum ω_m of natural frequencies ($m = 1, 2, \ldots$) and the corresponding eigenfunctions Φ_m form a complete and orthogonal system in the region Ω, bounded by the vessel walls and the plane $z = 0$. For simplicity, we will assume the frequencies to be nonmultiple. In the region D of the plane $z = 0$ bounded by the contour Γ, at $\sigma = 0$, we have instead of eqs.(5.1), (5.2)

$$i\omega_m F_m - \partial\Phi_m/\partial z = 0, \tag{5.3}$$

$$i\omega_m g^{-1}\Phi_m + F_m = 0. \tag{5.4}$$

Since the walls are vertical, it follows that $\partial\Phi_m/\partial N = \partial\Phi_m/\partial n = \partial F_m/\partial n = 0$ for the contour Γ. Note also that the functions $F_m(x, y)$ form a complete and orthogonal system in the region D. We subordinate these to the condition of norming

$$\int_D F_k F_l \, dx \, dy = \delta_{kl}. \tag{5.5}$$

The eigenfrequencies ω_m and the functions Φ_m, F_m of the nonperturbation problem ($\sigma = \varepsilon = 0$) will be assumed as known. The solution of the perturbation problem ($\varepsilon \neq 0$), close to the m-th natural oscillation in the nonperturbed case, will be sought by the method of the perturbation theory (Bibl.11), adopting /1090 $O(\varepsilon^2)$ as the sufficient accuracy:

$$\omega = \omega_m(1 + \varepsilon\lambda + \varepsilon^2\mu),$$
$$\Phi = \Phi_m + \sum_{k=1}^{\infty}(\varepsilon A_k + \varepsilon^2 B_k)\Phi_k, \qquad F = F_m + \sum_{k=1}^{\infty}(\varepsilon a_k + \varepsilon^2 b_k)F_k. \tag{5.6}$$

The Laplace equation, the condition at the walls for Φ, and the condition $\partial F/\partial n = 0$ for Γ are then satisfied. Equations (5.1), (5.2) must be removed to the region D of the plane xy, and eqs.(5.6) must be substituted there; all functions must be expanded in a series with respect to the functions of the orthonormed system F_m. Equating the coefficients for identical degrees of ε, we will find the required corrections for the frequency (λ, μ) and the eigenfunctions (A_k, B_k, a_k, b_k).

The Fourier coefficients of the function $g(x, y)$, determined in the region D, are calculated from the formula

$$g_k = \int_D g F_k \, dx \, dy. \tag{5.7}$$

If $\|g\| = o(\varepsilon^2)$, where the norm is construed as the space L_2, then $|g_k| = o(\varepsilon^2)$, a function which can be neglected in eqs.(5.1), (5.2). Note that since h, h_n are zero outside D_ε, while inside D_ε the estimates (4.10) apply [the area of the region D_ε being $O(\varepsilon)$], then $\|h\| \sim \varepsilon^2$, $\|h_n\| \sim \xi$; below, this will be taken into account.

2. Since the projection of the free surface onto the plane z = 0 coincides with the region D, eqs.(5.1), (5.2) may be canceled over the vertical to D.

For any function $f(x, y, z)$ on the free surface $z = h - h_0$, we have

$$f(x, y, z) = f(x, y, 0) + f_z(x, y, 0)(h - h_0) + \ldots, \tag{5.8}$$

where the norm of the discarded terms is smaller than $O(\varepsilon^2)$. Converting Φ, Φ_x, Φ_y, Φ_z according to eq.(5.8) and utilizing eqs.(4.20) and $\Phi_{zz} = -\Phi_{xx} - \Phi_{yy}$, we have instead of eqs.(5.1), (5.2),

$$[illegible] - \Phi_z = h_0\Delta\Phi - h\Delta\Phi - (\nabla h, \nabla\Phi) - h(\nabla h, \nabla\Phi_z) = Q_1, \tag{5.9}$$

$$\frac{i\omega}{g}\Phi + F = \frac{i\omega}{g}h_0\Phi_z + l^2\varepsilon^2\Delta F - \frac{i\omega}{g}h\Phi_z - \frac{3hh_n}{1 + h_n^2}\frac{\partial F}{\partial n} = Q_2. \tag{5.10}$$

Here, the terms with the norm $o(\varepsilon^2)$ are discarded and the operations ∇, Δ are calculated with respect to the variables x, y.

Since the walls are vertical, eqs.(5.3), (5.4), (5.6) yield, for the contour Γ,

$$\frac{\partial\Phi_m}{\partial n} = \frac{\partial F_m}{\partial n} = \frac{\partial\Phi_{mz}}{\partial n} = \frac{\partial F}{\partial n} = \frac{\partial\Phi_z}{\partial n} = \frac{\partial\Phi}{\partial n} = 0. \tag{5.11}$$

In the region D_ε, the derivatives (5.11) are $O(\varepsilon)$. Therefore, the last term in Q_2 from eq.(5.10) is of the order of ε^2 in D_ε and is zero outside D_ε, i.e., it can be discarded without impairing the accuracy.

Analogously, considering that in this case $h_s = 0$, $h_n \sim 1$ in D_ε, we conclude that $(\nabla h, \nabla\Phi_z) \sim \varepsilon$ in D_ε, so that the last term in Q_1 from eq.(5.9) also can be discarded. The other terms in Q_1, Q_2 have the norm $O(\varepsilon^2)$; in particular, $h_0 \sim \varepsilon^2$, as is evident from eq.(4.19). Hence, on substituting the series (5.6) into Q_1, Q_2, a substitution of only the main terms $F = F_m$, $\Phi = \Phi_m$, $\omega = \omega_m$ will suffice. On additionally expressing Φ_m, Φ_{mz} with the aid of eqs.(5.3), (5.4), we have /1091

$$Q_1 = -\frac{ig}{\omega_m}[h_0\Delta F_m - \mathrm{div}(h\nabla F_m)], \quad Q_2 = [illegible] - \frac{\omega_m^2}{g}h_0F_m + \frac{\omega_m^2}{g}hF_m. \tag{5.12}$$

Let us then calculate the Fourier coefficients of the functions Q_1, Q_2 according to eq.(5.7). First of all, on the basis of eq.(5.11), we have

$$\int_D F_k \Delta F_m \, dx\, dy = -\int_D (\nabla F_m, \nabla F_k)\, dx\, dy,$$
$$\int_D F_k \operatorname{div}(h \nabla F_m)\, dx\, dy = -\int_D h(\nabla F_m, \nabla F_k)\, dx\, dy. \tag{5.13}$$

An integration over the region D of the functions that are nonzero only in D_ε reduces to an integration over n from 0 to ∞ (h rapidly decreases when $n \to \infty$) and to an integration over the arc s of the contour Γ. Then, the functions F_m, ∇F_m can be replaced, without reducing the accuracy, by their values over Γ. Note also that, for $\delta = 0$, it follows from eq.(4.18) that

$$\int_0^\infty h\, dn = S = l^2 \varepsilon^2 \cos\gamma. \tag{5.14}$$

Using eqs.(5.13), (5.14), (5.5), we find the required Fourier coefficients of the functions Q_1, Q_2 from eq.(5.12) in the form of

$$Q_{1k} = -\frac{g}{\omega_m^2}\left[-h_0 \int_D (\nabla F_m, \nabla F_k)\, dx\, dy + l^2\varepsilon^2 \cos\gamma \oint_\Gamma \frac{\partial F_m}{\partial s}\frac{\partial F_k}{\partial s}\, ds\right],$$
$$Q_{2k} = -l^2\varepsilon^2 \int_D (\nabla F_m, \nabla F_k)\, dx\, dy - \frac{\omega_m^2}{g} h_0 \delta_{mk} + \frac{\omega_m^2}{g} l^2 \varepsilon^2 \cos\gamma \oint_\Gamma F_m F_k\, ds. \tag{5.15}$$

where h_0 is determined by eq.(4.19) for $\delta = 0$, γ = const, with L being the length of the contour Γ:

$$h_0 = \frac{l^2 \varepsilon^2 L \cos\gamma}{D}. \tag{5.16}$$

Since the Fourier coefficients Q_{1k}, Q_{2k} of the right-hand side of eqs.(5.9), (5.10) are $O(\varepsilon^2)$, for terms of the order of ε in the series (5.6) we have a homogeneous system of equations which is satisfied by a zero solution. In other words, since the perturbation is of the order of ε^2, the corrections for the eigenfrequencies and eigenfunctions also are of this order. Therefore, $\lambda = A_k = a_k = 0$, $k = 1, 2, \ldots$) in eq.(5.6).

Using eqs.(5.3), (5.4) we will transform the series (5.6) in the region D:

$$\omega = \omega_m(1+\varepsilon^2\mu), \qquad \Phi = \frac{ig}{\omega_m}F_m + ig\varepsilon^2\sum_{k=1}^{\infty}\frac{B_k}{\omega_k}F_k,$$
$$F = F_m + \varepsilon^2\sum_{k=1}^{\infty} b_k F_k, \qquad \Phi_t = i\omega_m F_m + i\varepsilon^2\sum_{k=1}^{\infty}\omega_k B_k F_k \tag{5.17}$$

Let us next substitute eq.(5.17) into the left-hand sides of eqs.(5.9), (5.10) and equate the Fourier coefficients of the left- and right-hand sides of these /1092 equations. This will yield the algebraic solutions

$$i\omega_m\varepsilon^2 b_m + i\varepsilon^2\mu\omega_m - i\omega_m\varepsilon^2 B_m = Q_{1m},$$
$$-\varepsilon^2\mu - \varepsilon^2 B_m + \varepsilon^2 b_m = Q_{2m} \tag{5.18}$$

for k = m and

$$i\omega_m\varepsilon^2 b_k - i\varepsilon^2\omega_k B_k = Q_{1k}, \qquad -\frac{\omega_m}{\omega_k}\varepsilon^2 B_k + \varepsilon^2 b_k = Q_{2k} \tag{5.19}$$

for $k \neq m$. From the systems (5.18), (5.19), we have

$$\varepsilon^2\mu = \frac{-iQ_{1m} - \omega_m Q_{2m}}{2\omega_m}, \qquad \varepsilon^2(B_m - b_m) = \frac{iQ_{1m} - \omega_m Q_{2m}}{2\omega_m},$$
$$\varepsilon^2 B_k = \frac{\omega_k(iQ_{1k} + \omega_m Q_{2k})}{\omega_k^2 - \omega_m^2}, \qquad \varepsilon^2 b_k = \frac{i\omega_m Q_{1k} + \omega_k^2 Q_{2k}}{\omega_k^2 - \omega_m^2} \tag{5.20}$$

Thus, the problem formulated for the case of vertical walls is solved. All the nonzero coefficients of the series (5.6) are expressed by means of eqs.(5.20), (5.15), (5.16) in the form of ω_k, F_k, contact angle γ, parameters of the vessel, and properties of the fluid. The coefficients B_m, b_m are, by virtue of eq.(5.20), determined with an accuracy to within an arbitrary term, which can be selected on condition of norming the perturbed eigenfunctions. Note that the corrections calculated in this case are proportional to ε^2, i.e., to σ.

In the particular case of $\gamma = \pi/2$, eqs.(5.15), (5.16) yield

$$h_0 = 0, \qquad Q_{1k} = 0, \qquad Q_{2k} = -k\varepsilon^2\iint_D (\nabla F_m, \nabla F_k)\,dx\,dy.$$

Here, the perturbation is determined by the integral with respect to the entire free surface (surface effect).

In the general case of $\gamma \neq \pi/2$, eqs.(5.15) contain also integrals over the contour Γ, expressing the boundary effect of the meniscus (pronounced curvature of the free surface at the walls). In the case of vertical walls, both effects (surface and boundary) are of the same order.

We will write the correction for the m-th eigenfrequency by using eqs.(5.20), (5.15), (5.16), and (4.9):

$$\frac{\omega-\omega_m}{\omega_m}=\frac{\sigma}{2\rho g}\left\{\int_D (\nabla F_m)^2\,dx\,dy+\frac{\omega_m^2 L\cos\gamma}{gD}\left[1-\frac{g^2}{\omega_m^4}\int_D (\nabla F_m)^2\,dx\,dy-\right.\right.$$
$$\left.\left.-\frac{D}{L}\oint_\Gamma F_m^2\,ds+\frac{g^2 D}{\omega_m^4 L}\oint_\Gamma\left(\frac{\partial F_m}{\partial s}\right)^2 ds\right]\right\}. \tag{5.21}$$

In the case of $\gamma = \pi/2$, when the static free surface is plane, eq.(5.21) will yield the formula

$$\frac{\omega-\omega_m}{\omega_m}=\frac{\sigma}{2\rho g}\int_D (\nabla F_m)^2\,dx\,dy>0, \tag{5.22}$$

showing that the eigenfrequency increases owing to surface tension, which is produced by the elasticity of the free-surface film. This case [in particular, eq.(5.22)] was considered by A.A.Petrov. At $\gamma \neq \pi/2$, eq.(5.21) contains terms associated with the boundary effect, which are proportional to $\cos\gamma$ and /1093 have different signs for wetting ($\gamma < \pi/2$) and nonwetting ($\gamma > \pi/2$) fluids. The total effect [sign of the entire right-hand side of eq.(5.21)] for $\gamma \neq \pi/2$ may differ.

Let us consider also the case of a two-dimensional motion of the fluid in the plane xz. In this case, $F = F(x)$ and the free boundary will be a line or, if $\sigma = 0$, a segment $[x_1, x_2]$ of the x-axis. In our calculations, and particularly in eqs.(5.5), (5.15), the integrals over the region D must be replaced by integrals over the segment $[x_1, x_2]$ and the contour integrals over Γ, by the sum of the values of the functions at the points $x = x_1$ and $x = x_2$. The terms containing derivatives to s will be discarded and the area D will be replaced by the length of the segment $x_2 - x_1$. Equations (5.20) will not change, but the formula for correction for eigenfrequency will become

$$\frac{\omega - \omega_m}{\omega_m} = \frac{\sigma}{2\rho g}\Big\{\int_{x_1}^{x_2}[F_m'(x)]^2\,dx + \frac{\omega_m^2}{g(x_2 - x_1)}[\cos\gamma(x_1) + \cos\gamma(x_2)]\times$$

$$\times\Big[1 - \frac{g^2}{\omega_m^4}\int_{x_1}^{x_2}[F_m'(x)]^2\,dx\Big] - \frac{\omega_m^2}{g}[\cos\gamma(x_1)F_m^2(x_1) + \cos\gamma(x_2)F_m^2(x_2)]\Big\}. \tag{5.23}$$

Equation (5.23) is valid also in the case where the contact angles differ for $x = x_1$ and $x = x_2$.

Section 6. Examples

Let us consider the oscillations of a liquid in a vessel with vertical walls, given a constant depth H. In this case, as is known (Bibl.8),

$$\Phi_m = \psi_m(x, y)\cosh k_m(z + H),$$

where the constant k_m is associated with the frequency ω_m:

$$\omega_m^2 = k_m g \tanh k_m H. \tag{6.1}$$

Laplace's equation for Φ_m and eqs.(5.11) show that F_m satisfies, in the region D, the boundary-value problem

$$\Delta F_m + k_m^2 F_m = 0 \text{ in } D, \qquad \partial F_m / \partial n = 0 \text{ in } \Gamma. \tag{6.2}$$

1. First, let us consider two-dimensional oscillations of the fluid (in the xz-plane) in a rectangular vessel with walls x = 0, x = a. The solution of the problem (6.2) with respect to eigenvalues in this case [the region D is here the segment [0, a] of the x-axis] will be

$$F_m = c_m \cos\frac{\pi m x}{a}, \quad c_m = \sqrt{\frac{2}{a(1 + \delta_{m0})}}, \quad k_m = \frac{\pi m}{a}, \quad m = 0, 1, 2, \dots. \tag{6.3}$$

Here, c_m is selected from the norming condition (5.5) and δ_{m0} is the Kronecker delta. Substituting eq.(6.3) into the general formula (5.23) with $\cos\gamma(0) =$ $= \cos\gamma(a)$, simple transformations on the basis of eq.(6.1) will yield

$$\frac{\omega - \omega_m}{\omega_m} = \frac{\sigma\pi^2 m^2}{2\rho g a^2}\left(1 - \frac{4\cos\gamma}{\pi m}\coth\frac{2\pi m H}{a}\right), \quad m = 1, 2, \ldots \tag{6.4}$$

If $\Pi \to 0$, the coefficient for $\cos\gamma$ in eq.(6.4) increases without limits. Therefore, for sufficiently small vessels, the correction for eigenfrequency is chiefly determined by the boundary effect and depends on the wettability of the walls: for the wettability ($\gamma < \pi/2$), this correction is negative and for /1094 the wettability ($\gamma > \pi/2$), positive. If, on the other hand, $m \to \infty$ for fixed a, H, then the coefficient for $\cos\gamma$ in the parentheses of eq.(6.4) decreases. Therefore, given sufficiently high frequencies, the frequency correction is determined by the surface effect, is positive, and rapidly increases with increasing m. For sufficiently large m, this correction becomes so large that the postulates of the perturbation theory, utilized above, no longer are applicable.

Of greatest interest is the correction for the lower nonzero eigenfrequency ω_1 (m = 1). If the fluid does not wet the walls ($\gamma > \pi/2$), the frequency ω_1 always will increase. In the case of a wetting fluid ($\gamma < \pi/2$), the correction for ω_1 may have either sign. If $\cos\gamma > \pi/4$ or $\gamma < 38^\circ$, the lower frequency will decrease ($\omega < \omega_1$), as follows from eq.(6.4), for any a, H.

2. Let us then consider the oscillations of a fluid in a cylindrical vessel of a depth H and a radius R. The eigenfunctions of the problem (6.2) have the following form for a circle of radius R (Bibl.12):

$$\begin{gathered} F_{ms}^{(1)} = c_{ms} J_m(\mu_{ms} r/R) \cos m\theta, \quad F_{ms}^{(2)} = c_{ms} J_m(\mu_{ms} r/R) \sin m\theta, \\ F_{0s} = c_{0s} J_0(\mu_{0s} r/R), \quad F_{00} = c_{00}, \quad m, s = 1, 2, \ldots \end{gathered} \tag{6.5}$$

where r, θ are polar coordinates in the xy-plane, with the center at the cylinder axis, μ_{ms} are positive consecutive roots of the derivatives of Bessel functions

$$J_m'(\mu_{ms}) = 0, \quad 0 < \mu_{m1} < \mu_{m2} < \ldots, \quad m = 0, 1, 2, \ldots, \quad s = 1, 2, \ldots$$

The numbers k_{ms} are related to μ_{ms} by $k_{ms} R = \mu_{ms}$ for $m = 0, 1, 2, \ldots$, $s = 1, 2, \ldots$, and $k_{00} = 0$. The frequencies ω_{ms} are expressed through k_{ms} by the common formula (6.1). From the norming condition (5.5) for the functions (6.5) we have (Bibl.12)

$$c_{00} = \frac{1}{\sqrt{\pi} R}, \qquad c_{ms} = \frac{\sqrt{2}\,\mu_{ms}}{\sqrt{\pi} R \sqrt{1 + \delta_{m0}} \sqrt{\mu_{ms}^2 - m^2}\,|J_m(\mu_{ms})|}, \qquad m = 0, 1, 2, \ldots, \quad s = 1, 2, \ldots \tag{6.6}$$

where the frequencies ω_{ms} are double when $m \geq 1$ and then correspond to the eigenfunctions $F_{ms}^{(1)}$, $F_{ms}^{(2)}$ from eq.(6.5). For multiple frequencies (degenerate case) the method of the perturbation theory used in Section 5 is, generally speaking, modified and the multiple frequencies are split. In this case, the multiplicity of the eigenfrequencies of the cylindrical vessel is due to its axial symmetry. Since the perturbation also is symmetric (γ = const), the eigenfrequencies of the perturbation problem remain multiple (perturbation does not exclude degeneration). Therefore, the corrections here may be calculated according to the previous formulas. Substituting eqs.(6.5), (6.6) into eq.(5.21) will yield

$$\frac{\omega - \omega_{ms}}{\omega_{ms}} = \frac{\text{[illegible]}}{\text{[illegible]}\rho g R^2} \left\{ \frac{2\cos\gamma}{\mu_{ms}(\mu_{ms}^2 - m^2)} \left[(\mu_{ms}^2 - \text{[illegible]}\, H/R) + \text{[illegible]} + m^2 \operatorname{th}(\mu_{ms} H/R)\right] \right\}, \quad m = 0, 1, 2, \ldots, \quad s = 1, 2, \ldots \tag{6.7}$$

As in the case of a rectangular vessel [eq.(6.4)] for small H, the principal role in eq.(6.7) is played by the terms associated with the boundary effect and for large m, s - by the terms associated with the surface effect. The lower nonzero eigenfrequency corresponds to the least number μ_{ms}, equal to /1095

$$\mu_{11} = 1.841, \quad m = 1, \quad s = 1.$$

In the presence of this natural oscillation, the points of the free surface lying on some diameter of the circle $r = R$ remain fixed. The correction for the frequency ω_{11} will be positive for a nonwetting fluid, whereas for a wetting fluid it can be of either sign. This correction will be negative for any H, R provided that $M \cos\gamma > 1$, where M is the minimum - with respect to H, R - of the coefficient for $\cos\gamma$ within the braces in eq.(6.7) at $m = 1$, $s = 1$. It is readily shown that this minimum is attained at $H = \infty$, and eq.(6.7) then can be simplified to

$$\frac{\omega - \omega_{ms}}{\omega_{ms}} = \frac{\sigma\mu_{ms}^2}{2\rho gR^2}\left(1 - \frac{2\cos\gamma}{\mu_{ms}}\right).$$

Therefore, at $\cos\gamma > \mu_{11}/2 \approx 0.92$, $\gamma < 23^\circ$ the lower frequency of the natural oscillations of a fluid in a cylindrical vessel decreases owing to surface tension (no matter what the depth and radius of the vessel).

The authors wish to express their gratitude to A.D.Myshkis and A.D.Tyuptsov for reading the manuscript of this article and furnishing comments which were used in final editing.

BIBLIOGRAPHY

1. Landau, L.D. and Lifshits, Ye.M.: Solid-State Mechanics (Mekhanika sploshnykh sred). Moscow, Gostekhizdat, 1953.

2. Gobson, Ye.V. (Hobson, E.V.): Theory of Spherical and Ellipsoidal Functions (Teoriya sfericheskikh i ellipsoidal'nykh funktsiy). Moscow, Izd. Inostr. Lit., 1952.

3. Moiseyev, N.N.: Introduction to the Theory of Oscillations of Liquid-Containing Bodies. Advan. Appl. Mech., Vol.8, pp.230-295, 1964.

4. Mikhlin, S.G.: Variational Methods in Mathematical Physics (Variatsionnyye metody v matematicheskoy fizike). Moscow, Gostekhizdat, 1957.

5. Moiseyev, N.N.: On Certain Mathematical Problems of the Relative Motion of Satellites (Sur certains problèmes mathématiques du mouvement relatif des satellites). Dynamics of Satellites (Symposium, Paris, May 1962). Springer-Verlag, Berlin, pp.313-355, 1963.

6. Chernous'ko, F.L.: Motion of a Solid with a Cavity Containing an Ideal Fluid and an Air Bubble (O dvizhenii tverdogo tela s polost'yu, soderzhashchey ideal'nuyu zhidkost' i puzyr' vozdukha). Prikl. Matem. i Mekhan., Vol.28, No.4, pp.735-745, 1964.

7. Moiseyev, N.N.: Narrow-Band Type Asymptotic Methods. In Coll.: Certain Problems of Mathematics and Mechanics (Asimptoticheskiye metody tipa uzkikh polos. V sb. Nekotoryye problemy matem. i mekhan.). Novosibirsk, Izd. Sibirsk. Otd. Akad. Nauk SSSR, pp.180-200, 1961.

8. Kochin, N.B., Kibel', I.A., and Roze, N.V.: Theoretical Hydromechanics (Teoreticheskaya gidromekhanika). Vol.I, II, Moscow, Fizmatgiz, 1963.

9. Vishik, M.I. and Lyusternik, L.A.: Regular Degeneration and the Boundary Layer for Small-Parameter Linear Differential Equations (Regulyarnoye vyrozhdeniye i pogranichnyy sloy dlya lineynykh differentsial'nykh uravneniy s malym parametrom). Usp. Matem. Nauk, Vol.12, No.5(77), pp.3-122, 1957.

10. Smirnov, V.I.: Course in Higher Mathematics (Kurs vysshey matematiki). Vol.IV, Moscow, Fizmatgiz, 1958.

11. Titchmarsh, E.Ch.: Expansions of Eigenfunctions Associated with Second-Order Differential Equations (Razlozheniya po sobstvennym funktsiyam, svyazannyye s differentsial'nymi uravneniyami vtorogo poryadka). Vol.II,

Moscow, Izd. Inostr. Lit., 1961.

12. Babich, V.M. et al.: Linear Equations of Mathematical Physics (Lineynyye uravneniya matematicheskoy fiziki). Moscow, Izd. Nauka, 1964.

Received July 30, 1964